OUR HERITAGE
ISSN: 0474-9030 Vol-68, Special Issue-6
Impact Factor (2020) - 6.8
Special Issue on *"Importance of Mathematics Education"*

Existence and Uniqueness of Solution of Second Order Nonlinear Ordinary Differential Equations

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Abstract

In this paper we have discussed the study of existence and uniqueness of solutions of nonlinear second order ordinary differential equations with initial conditions. In first section, we prove existence and uniqueness by using Picard's Theorem. In next section we prove existence and uniqueness by using Lipschitz Condition.

Keywords: nonlinear, existence, uniqueness, Lipschitz,

Introduction:

Nonlinear Ordinary Differential Equation

An Ordinary Differential Equation $z'(x) = f(x, u(x))$ is called nonlinear if and only if the function f is nonlinear in the second argument.
Example:
(i)The Differential Equation $z'(x) = \frac{x^2}{z^3(x)}$, is nonlinear, because the function

$f(x, u) = \frac{x^2}{u^3}$ is nonlinear in the second argument.
(ii)The Differential Equation $z'(x) = 2xz(x) + \log(z(x))$ is nonlinear, because the function $f(x, u) = 2xu + \log(u)$is nonlinear in the second argument, because of the term $\log(u)$.

EXISTENCE AND UNIQUENESS : PICARD'S THEOREM

Consider a continuous function $f: \mathrm{I} \times \mathrm{R}$ whereI be a Real Interval, f satisfies the Lipschitz condition i.e.

$$|f(y, z_2) - f(y, z_1)| \leq K\,|z_2 - z_1|$$

Suppose $y_0 \in$ I. Then there exists a unique function z on I s.t.

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$$z''(y) = f[y, z(y), z'(y)] \quad \text{for } y \in I$$

with initial conditions $z(y_0) = z_0$, $z'(y_0) = z_0{}'$

Note : By the Mean-Value -Theorem, the above function satisfies the Lipschitz condition if $\frac{\partial^2}{\partial y^2} f[y, z(y), z'(y)] \leq K$ on I × R.

LEMMA 1 :

Here we show the result is true for the case $y_0 = z'_0 = 0$.

Proof. Let us suppose that the result is known in this case ,y_0, z_0 are given.

Suppose $g(y, z) = f(y + y_0, z' + z'_0)$, by assumption, there is a unique function z s.t.

$$z''(y) = g\,[y, x(y), x'(y)] \quad \text{when } y + y_0 \in I$$

And $x(0) = 0$. Let $z''(y) = x''(y - y_0) + z'_0$. Then $z'(y_0) = z'_0$ and for $y \in I$,

$$z''(y) = x''(y - y_0)$$

After that here we show the properties of z'' by expressing it into an integral form.

LEMMA 2 :

Let I∈0, for a derivable function on I, then the following are equivalent.

(i) $z''(y) = f[y, z(y), z'(y)]$ for $y \in I$ and $z(y_0) = z_0$, $z'(y_0) = z_0{}'$;
(ii) $z''(y) = \int_0^y f[t, z(t), z'(t)]dt$ for $y \in I$.

Proof . (i) ⇒ (ii)

Suppose that C(I) is the space of all continuous functions on I, we defines a mapping A : C(I) → C(I)as follows : for z ∈ C(I)

$$(Az')(y) = \int_0^y f[t, z(t), z'(t)]dt.$$

By the assumption ,z' is the required solution iff $A(z') = z'$.

We take

$$z'_0 = 0, z'_1 = A(z'_0), z'_2 = A(z'_1), \ldots\ldots, z'_n = A(z'_{n-1}) = An(z'_0).$$

Here we prove the uniform convergence of $\{z'_n\}$ to a limit function z', then

$A(z') = \lim_{n\to\infty} A(z'_n) = \lim_{n\to\infty} z'_{n+1} = z'$ as required.

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LEMMA 3 :

Suppose that $|u(y) - v(y)| \leq C$ for $y \in I$. Then for each $n \geq 1$,

$$|(A^n u)(y) - (A^n v)(y)| \leq C\frac{(K|y|)^n}{n!} \text{for } y \in I.$$

Proof. By the Lipschitz condition,

$|f[t, u(t)] - f[t, v(t)]| \leq CK$ for $t \in I$, so

$$|(Au)(y) - (Au)(y)| = \left|\int_0^y f(t, u(t)) - f(t, v(t))\right| dt$$

$$\leq CK\,|y|,$$

that is the statement true for n = 1.Suppose it is true for any n, then by the Lipschitz condition,

$$|f[t, (A^n u)(t)]| - f[t, (A^n v)(t)] \leq KC\frac{(K|t|)^n}{n!}.$$

So for $y > 0$;

$$|(A^{n+1}u)(y) - (A^{n+1}v)(y)| = \left|\int_0^y \{f[t(A^n u)(t)] - f[t, (A^n v)](t)\}\, dt\right|$$

similarly for $y < 0$, with $|y|$, instead of y.

Proof of the Theorem:

Consider that I is bounded and closed interval.

Then $|y| \leq R$ for $y \in$I, and continuous functions are bounded on I, so there is M s.t.$|f(t,0)| \leq$ M for $t \in$I.

Uniqueness :

Let us suppose that $Au = u$ and $Av = v$,

then $A^n u = u$ and $A^n v = v$ for all n.

Again, there exists C such that $|u(y) - v(y)| \leq C$ for $y \in I$.

By LEMMA 3,

$$|u(y) - v(y)| \leq C\frac{(K|y|^n)}{n!} \text{for all } n \geq 1.$$

But for every y, $K|y|)^n/n! \to 0$ as $n \to \infty$. Therefore $u(y) = v(y)$.

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Existence :

Let $z'_0 = 0, z'_1 = A(z'_0), z'_2 = A(z'_1), \ldots\ldots, z'_n = A^n(z'_0)$, then

$$|z'_1(y) - z'_0(y)| = |z'_1(y)| = \int_0^y f(t,0)dt\,M \leq |y| \leq C$$

For $y \in I$. Now $\sum_{n=1}^{\infty}(KR)^n/n!$ is convergent *to* e^{KR}, hence by the M-test,

$\sum_{n=1}^{\infty}(z'_{n+1} - z'_n)$ converges uniformly to y on I .

But $$(z'_1 - z'_0) + (z'_2 - z'_1) + \ldots\ldots + (z'_n - z'_{n-1}) = z'_n - z'_0 = z'_n,$$

hence $z'_n \to z'$ uniformly on I i.e.$|z'_n(y) - z'(y)| \leq \delta_n$ for $y \in I$, where

$\delta_n \to 0$ as $n \to \infty$.In the case $n = 1$, in LEMMA 3

$$|z'_{n+1}(y) - (Az')(y)| = |(Az'_n)(y) - Az'(y)| \leq K|y|\delta_n$$

for $y \in I$, hence

$$(Az')(y) = \lim_{n\to\infty} z'_{n+1}(y) = z'(y)$$

Now, consider I is an open or unbounded interval. We express it as $\cup_{n=1}^{\infty} I_n$ where $I_1 \subset I_2 \subset I_3 \ldots$. are bounded and closed intervals with $0 \in I_1$. In previous, for every n,a unique solution $z'_{(n)}$ on I_n with $z'_{(n)}(0) = 0$, by this Uniqueness, $z'_{(n+1)}$ agrees with $z'_{(n)}$ on I_n, hence it is consistent to define z' on I by ,

$$z'(y) = z'_{(n)}(y), \qquad \text{for } y \in I_n.$$

hence, y satisfies the equation on I.

Here we give two examples which shows that , if the Lipschitz condition is not satisfied then the theorem fails.

EXAMPLE:

1.Consider the Initial-Value-Problem $z'' = -z'^2$, with $z'(1) = 1$, on the interval $[-1, 1]$.

Solving by elementary methods, we have

$z'^2 \ldots 1,$

$\frac{1}{z'} = y + \ldots$

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Existence :

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$$|z'_1(y) - z'_0(y)| = |z'_1(y)| = \int_0^y f(t,0)dt\, M \leq |y| \leq C$$

For $y \in I$. Now $\sum_{n=1}^{\infty}(KR)^n/n!$ is convergent *to* e^{KR}, hence by the M-test,

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hence $z'_n \to z'$ uniformly on I i.e. $|z'_n(y) - z'(y)| \leq \delta_n$ for $y \in I$, where

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$$|z'_{n+1}(y) - (Az')(y)| = |(Az'_n)(y) - Az'(y)| \leq K|y|\delta_n$$

for $y \in I$, hence

$$(Az')(y) = \lim_{n\to\infty} z'_{n+1}(y) = z'(y)$$

Now, consider I is an open or unbounded interval. We express it as $\cup_{n=1}^{\infty} I_n$ where $I_1 \subset I_2 \subset I_3 \ldots$ are bounded and closed intervals with $0 \in I_1$. In previous, for every n, a unique solution $z'_{(n)}$ on I_n with $z'_{(n)}(0) = 0$, by this Uniqueness, $z'_{(n+1)}$ agrees with $z'_{(n)}$ on I_n, hence it is consistent to define z' on I by ,

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hence, y satisfies the equation on I.

Here we give two examples which shows that , if the Lipschitz condition is not satisfied then the theorem fails.

EXAMPLE:

1. Consider the Initial-Value-Problem $z'' = -z'^2$, with $z'(1) = 1$, on the interval $[-1, 1]$.

Solving by elementary methods, we have

$$\frac{z''}{z'^2} = -1,$$

$$\frac{1}{z'} = y + \text{[illegible]}$$

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The initial condition $z'(1) = 1$,gives the solution $z' = 1/y$. This is the unique solution on $(0,1]$. It cannot be extended to a differentiable function at 0, so there is no solution on $[-1,1]$. To see that the Lipschitz condition fails, note that $f(y,z,z') = -z'^2$.

We have

$$f(y,z'+1) - f(y,z,z') = -(z'+1)^2 + z'^2 = -2z' - 1,$$

which is not bounded on R.

2. Consider the initial value problem

$z'' = 3z'^{\frac{2}{3}}$ with $z'(0) = 0$, on $I = R$

here, 0 is one solution, elementary methods gives

$$\frac{1}{3z'^{2/3}} z'' = 1$$

$$z'^{1/3} = y + c,$$

$$z' = (y + c)^3.$$

Hence y^3 is another solution with $z'(0) = 0$. There are infinitely many solutions, for each c > 0, a solution is

$$z_c'(y) = \begin{cases} (y-c)^3 & for\ y \geq c \\ 0 & for\ y < c \end{cases}$$

Then z'_c is differentiable at c, with derivative 0. The Lipschitz condition fails because

$$\frac{z'^{2/3} - 0}{z' - 0} = \frac{1}{z'^{1/3}} \to \infty \text{ as } z' \to 0^+ .$$

Conclusion: With the help of Picard's Theorem we proved the existence and uniqueness of solution of nonlinear differential equation with uniform convergence. Here we conclude that , if the Lipschitz condition is not satisfied then the theorem fails.

References:

[1] Alka S.Kausadikar et al, Journal of Global Research in Mathematical Archives, Volume 2, No.4, April-2014, pages 79-82 .
[2] http://www.maths.lancs.ac.uk/jameson/picard.pdf
[3] https://www.rose-hulman.edu/bryan/lottamath/uniqde.pdf
[4] http://www.math.uiuc.edu/tyson/existence.pdf
[5] Campbell Stephen L, An Introduction to Differential Equations and Their Application, Longman (New York and London),(1986).
[6] Earl D. Rainville and phillip E. Bedientm, A short course in Differential Equation (Fourth Edition) The macmillan company, 1969, chapter 15 and 16w.

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